The Swing Ratio and Game Theory

DAVID SANKOFF
University of Montreal
AND
KOULA MELLOS
University of Ottawa

It is well known that in electoral systems where a single representative is elected from each district or constituency on the basis of a simple plurality, the proportion of seats in the legislature a party can expect to win is not equal to the proportion of the overall vote it receives. In a two-party system the minority party can expect to be underrepresented, while the party attracting more than 50 per cent of the popular vote usually wins a higher proportion of seats than votes. One indicator of this tendency is the *swing ratio* s, the change in representation per change in popular vote from one election to the next. If r is the proportion of seats a party wins, and p the proportion of the popular vote it receives, $s=\Delta r/\Delta p$, a ratio which has been found to be more or less constant in any one political context as long as p does not deviate too much from the 40–60 per cent range. Predictive models of the relationship between r and p may allow for a degree of statistical fluctuation. Such models predict $\bar{r}$ as a function of p, where $\bar{r}=E(r)$, the expected value of r. Then the quantity of interest is the expected swing ratio, $\bar{s}=E(s)=d\bar{r}/dp$, evaluated at $p=0.5$, i.e., where both parties have 50 per cent of the vote.

In Britain, it has been found[1] that $s\approx 3$; in the U.S.,[2] s ranges between 2.5 and 3.3 depending on the mode of calculation and also on the legislative body (House or Senate) being considered. For Quebec, data from 27 elections are best fitted by an s-value of 3.8. One suggestion, that $\bar{s}=3.0$ is characteristic of all political contexts with this type of electoral system, has been formulated in terms of the 'cube law', $\bar{r}=p^3/[p^3+(1-p)^3]$. Unfortunately, no large-scale systematic surveys have been made to test this hypothesis. Nevertheless, it seems clear that s ranges between 2 and 4 and not between 1.0 and 1.2, say, nor between 8 and 16.

The question which has not been satisfactorily answered, and to which this paper is addressed, is this: Why does s tend to take on values only in this particular range, independent of country and time? Kendall and Stuart[3] have shown that any sampling explanation based on the unevenness of distribution of party supporters among constituencies must involve quantitative assumptions which are just as difficult to explain as the swing ratio effect![4] March[5] proposed a reasonable model based on interparty negotiations and intra-constituency pressures; but to quantify this model would require assumptions about the values of a number of parameters. Furthermore the behavior of this model would tend to be rather sensitive to the values chosen for these parameters.

In this paper we define a two-person zero-sum game which is perhaps the simplest possible abstraction of this type of electoral contest. The idea is to try to account for all or part of the swing ratio effect in terms of resource allocation strategies by the two competing parties. There are no parameters to specify, and, as we shall show, under asymptotically optimal strategies, $\bar{s}\approx 2$.

The Game

There are two players, I and II, with resources θ and $1-\theta$, respectively, where $0<1-\theta<\theta<1$. There are n boxes and each player must allocate, secretly, his resources among the boxes. After this allocation is completed, the amount I put in each box is compared with the amount II put in, and whoever put more, wins the box. The payoff is the proportion of boxes won by I. Player I tries to maximize the minimum payoff he can expect for any strategy II may play; while II tries to minimize the maximum payoff expected for any strategy available to I.

The particular case of this game where $n=3$ was discussed by Owen,[6] but only for θ fixed at 0.5. The general case has not been discussed previously.

This game does not have a saddle point in pure strategies, except for the trivial case $\theta/n>1-\theta$. We prove this in the next section and show that

[1] M. G. Kendall and A. Stuart, "The Law of the Cubic Proportion in Election Results," *British Journal of Sociology*, 1 (September, 1950) 183–197.

[2] J. G. March, "Party Legislative Representation as a Function of Election Results," *Public Opinion Quarterly*, 21 (Winter, 1957–58), 521–542.

[3] Kendall and Stuart, "The Law of the Cubic Proportion . . ."

[4] A recent study which assumes the cube law is Henri Theil, "The Cube Law Revisited," *Journal of the American Statistical Association*, 65 (September, 1970) 1213–1219.

[5] March, "Party Legislative Representation . . ."

[6] G. Owen, *Game Theory* (Philadelphia: W. B. Saunders, 1968), pp. 88–93.

even as $n \to \infty$, i.e., for large numbers of constituencies, the game has no solution or approximate solution in pure strategies. Extending the game to allow mixed strategies, we show how to construct strategies depending on θ and n in a way which is asymptotically optimal as $n \to \infty$.

Following this analysis we devote a section to its interpretation in electoral terms.

Mathematical Analysis[7]

THEOREM 1. For any pure strategy available to II, player I has an opposing pure strategy producing a payoff of 1. For any pure strategy available to I, II has an opposing one producing a payoff less than or equal to $1-(1-\theta)/\theta+1/n$.

Proof. Let $y=(y_1, \cdots, y_n)$ be any pure strategy available to II; i.e., y_1 is allocated to the first box, y_2 to the second, etc. For $i=1, \cdots, n$, set $x_i=y_i+(2\theta-1)/n$. Then $x=(x_1, \cdots, x_n)$ is a strategy for I because

$$\sum_{i=1}^{n} x_i = \theta,$$

and the payoff for x against y is 1 since $x_i>y_i$ for $i=1, \cdots, n$.

Now let x be any pure strategy available to I. Without loss of generality we can assume

$$x_1 \leq x_2 \leq \cdots \leq x_n. \tag{1}$$

Let m be such that

$$\sum_{i=1}^{m} x_i < 1-\theta, \qquad \sum_{i=1}^{m+1} x_i \geq 1-\theta. \tag{2}$$

Clearly $m \geq 1$ unless $\theta/n \geq 1-\theta$, in which case the theorem is trivially true. Otherwise set

$$y_i = \begin{cases} x_i + \left(1-\theta-\sum_{j=1}^{m} x_j\right) \Big/ m, & \text{for } i = 1, \cdots, m \\ 0, & \text{for } i = m+1, \cdots, n. \end{cases}$$

The payoff for x against $y=(y_1, \cdots, y_n)$ is $1-m/n$. An upper bound for this quantity can be found as follows: From (1) and the second inequality of (2), we have

$$x_{m+1} \geq \frac{1-\theta}{m+1}.$$

By (1), this inequality must also be satisfied by each of $x_{m+2}, \cdots, x_n$. Thus

$$[n-(m+1)]\frac{1-\theta}{m+1} \leq \sum_{i=m+2}^{n} x_i. \tag{3}$$

But from (2),

$$\sum_{i=m+2}^{n} x_i \leq \theta-(1-\theta). \tag{4}$$

Inequalities (3) and (4) together yield

$$n\frac{1-\theta}{m+1}-(1-\theta) \leq \theta-(1-\theta),$$

$$1-m/n \leq 1-(1-\theta)/\theta+1/n.$$

This theorem shows that the game with pure strategies has no saddle point (except for trivial cases) and that when this game is played with many boxes, it is as far away from having a solution as when it is played with a few.

In contrast to this, we show in Theorem 2 how to construct a pair of *mixed* strategies which come within $3/2n$ of constituting a solution. For large n, such as encountered in electoral systems (say 100–1000), these strategies specify, for all practical purposes, a value for the game. Before proving this theorem we require some definitions.

For pure strategies x versus y the payoff to I will be denoted

$$A(x, y) = (1/n)\sum_{i=1}^{n} \chi(x_i, y_i),$$

where

$$\chi(a, b) = \begin{cases} 1, & \text{if } a > b \\ \frac{1}{2}, & \text{if } a = b \\ 0, & \text{if } a < b. \end{cases} \tag{5}$$

A mixed strategy for a player may be considered as a probability distribution over all the strategies available to him. Thus we work with the set of cumulative distribution functions on the n-dimensional interval $U=[0, 1]^n$ which concentrate all their mass on the hyperplane defined by

$$\sum_{i=1}^{n} x_i = \theta$$

in the case of player I, or

$$\sum_{i=1}^{n} y_i = 1-\theta$$

[7] We would like to thank Professors Mark Kac, Jean-Jacques Moreau, and Frank Stenger for advice, and Professor Guillermo Owen for information about this type of game. As far as possible we shall keep to the terminology and notation of Owen, *Game Theory*.

in the case of II. Since we are using mixed strategies, we are interested not just in the payoff, but in the *expected* payoff

$$E(F, G) = \int_U \int_U A(x, y) dF(x) dG(y) \quad (6)$$

where I plays mixed strategy F and II plays G. If G assigns probability 1 to the pure strategy y

$$E(F, G) = E(F, y) = \int_U A(x, y) dF(x)$$

and an analogous remark may be made about $E(x, G)$. The values of the game[8] for I and II are v_I and v_{II} respectively where

$$\sup_F \inf_y E(F, y) = v_I \leq v_{II} = \inf_G \sup_x E(x, G).$$

THEOREM 2.

$$v_I + 1/2n \geq (3\theta - 1)/2\theta \geq v_{II} - 1/n.$$

Proof. Let F be the strategy for I which assigns probability $1/n!$ to each permutation of the vector $(u, 3u, \ldots, [2n-1]u)$ where $u = \theta/n^2$. Let H be any mixed strategy for II. From (5) and (6)

$$E(F, H) = (1/n) \int_U \int_U \sum_{i=1}^n \chi(x_i, y_i) dF(x) dH(y) \quad (7)$$

$$= (1/n) \int_U \sum_{i=1}^n \sum_{j=1}^n (n-1)! \cdot \chi([2j-1]u, y_i)/n! dH(y)$$

since F assigns probability $1/n!$ to $n!$ different vectors, $(n-1)!$ of which have $[2j-1]u$ as the i-th coordinate, for each $j = 1, \ldots, n$. Then

$$E(F, H) = (1/n^2) \int_U \sum_{i=1}^n \sum_{j=1}^n \chi([2j-1]u, y_i) dH(y) \quad (8)$$

$$\geq (1/n^2) \int_U \sum_{i=1}^n (n - (y_i + u)/2u) dH(y), \quad (9)$$

[8] Optimal strategies exist if

$$\max_F \min_y E(F, y) = v = \min_G \max_x E(x, G),$$

in which case v is called the value of the game.

since $\chi([2j-1]u, y_i) = 1$ except when $y_i \geq [2j-1]u$; i.e., except for $j = 1, \ldots, r$ where r is the largest integer in $(y_i + u)/2u$. For H to be a strategy for II, it must concentrate all its mass on the hyperplane

$$\sum_{i=1}^n y_i = 1 - \theta,$$

where the integrand in (9) becomes $n^2 - (1-\theta)/2u - n/2$. Therefore

$$E(F, H) \geq (1 - (1-\theta)/2n^2u - 1/2n) \int_U dH(y) \quad (10)$$

$$= (3\theta - 1)/2\theta - 1/2n,$$

which is a lower bound for v_I, since H was chosen arbitrarily.

Similarly, let G assign probability $(n-m)!/n!$ to each distinct permutation of $(w, 3w, \ldots, [2m-1]w, 0, \ldots, 0)$ where $w = (1-\theta)/m^2$ and m is the largest integer in $n(1-\theta)/\theta$. Let H be any mixed strategy for I. As in (7)–(10),

$$E(H, G) = (1/n^2) \int_U \sum_{i=1}^n \left[\sum_{j=1}^m \chi(x_i, [2j-1]w) + \sum_{j=m+1}^n \chi(x_i, 0) \right] dH(x)$$

$$\leq (1/n^2) \int_U \sum_{i=1}^n [(x_i + w)/2w + n - m] dH(x)$$

$$= 1 + m[2n - m\theta/(1-\theta)]/2n^2 + 1/2n.$$

Using the fact that $n(1-\theta)/\theta - 1 < m \leq n(1-\theta)/\theta$, it follows that

$$E(H, G) \leq (3\theta - 1)/2\theta + 1/n,$$

and this completes the proof.

Electoral Interpretation

Theorem 2 shows that in a game with many boxes, if I and II are playing nearly optimal strategies (or optimal strategies if these exist), the average payoff will be approximately $(3\theta-1)/2\theta$.

How can this game be related to electoral systems? The n boxes, of course, represent n constituencies, and the payoff represents the proportion of these constituencies won by the majority party. The simplest way to interpret the resource variable θ is to equate it to p, the proportion of the total popular vote won by the majority party. This leads immediately to the prediction $\bar{r} = (3p-1)/2p$, and $\bar{s}$, evaluated at $p = 0.5$, is 2. Thus, if the parties in a two-party system could freely allocate their total support among the constituencies, we should expect a swing ratio equal to 2.

This model is, of course, excessively simple. It would be preferable to relate the variable θ to the sum total of electoral resources: financial, organizational, press goodwill, etc. Some of these resources can be freely allocated; others, however, are bound to certain constituencies. If we were somehow able to identify θ with the freely allocatable resources, there would still remain the problem of predicting the vote in each constituency. Part of the resources bound to a particular constituency is the proportion of hard-core voters. A realistic model would have to combine[9] the hard-core vote, which is relatively independent of free resources allocated, with the vote of the previously uncommitted constituents, which is more proportional to free resources allocated to the constituency.

[9] Thiel, "The Cube Law Revisited," shows one way of doing this.

It is easy to see how such a model would have a higher swing ratio than the pure game-theory model. Assuming the hard-core vote is $(1-z)/2$ for both parties in most constituencies, the game would be played to capture a majority of the uncommitted vote in as many constituencies as possible. Then $\bar{r}=(3\theta-1)/2\theta$ as before, but instead of $p=\theta$, we have $p=\theta z+(1-z)/2$, and hence $\bar{s}=2/z$ when $\theta=p=1/2$. So if one third of the voters, say, are hard-core, $\bar{s}=3$.

Parameters other than z, perhaps related to differences between constituencies or to the turnout proportion, could also be added to the game. Some of these would tend, like z, to amplify the expected swing ratio, and others would tend to attenuate it. The important point about the game model is that it gives a reasonable basic value of $\bar{s}$ without the introduction of *any* parameters. It is a very simple model. Nevertheless, we feel that it summarizes the strategic problem faced by parties in this type of electoral system and that the asymptotic solution captures the most important component of the swing ratio effect.