An Inequality for Probabilities
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AN INEQUALITY FOR PROBABILITIES

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1. The main result and applications. Given a probability measure space $(\Omega, \mathfrak{F}, P)$, let $A_k \in \mathfrak{F}$, $k=1, \cdots, N$. The main result is given in the following theorem.

THEOREM 1.1.

$$P\left(\bigcup_{k=1}^{N} A_k\right) \geqq \frac{\theta \mathcal{B}^2}{2\alpha + (2-\theta)\mathcal{B}} + \frac{(1-\theta)\mathcal{B}^2}{2\alpha + (1-\theta)\mathcal{B}} \tag{1.1}$$

where $\mathcal{B} = \sum_{k=1}^{N} P(A_k)$, $\alpha = \sum_{k=1}^{N} \sum_{l=1}^{k-1} P(A_k \cap A_l)$ *and* $\theta = 2\alpha/\mathcal{B} - [2\alpha/\mathcal{B}]$, $0 \leqq \theta < 1$.

The proof of Theorem 1.1 is given in §2.

COROLLARY 1. *A necessary condition for* $P(\bigcup_{k=1}^{N} A_k) < 1$ *is that* $\mathcal{B} < (1+(1+8\alpha)^{1/2})/2$.

PROOF. It is easy to verify that if the right-hand side of inequality (1.1) is regarded as a function of θ, then the minimum occurs for $\theta = 0$. Hence

$$P\left(\bigcup_{k=1}^{N} A_k\right) \geqq \frac{\mathcal{B}^2}{2\alpha + \mathcal{B}}.$$

Therefore if $1 > P(\bigcup_{k=1}^{N} A_k)$, then $1 > \mathcal{B}^2/(2\alpha + \mathcal{B})$, that is, $\mathcal{B}^2 - \mathcal{B} - 2\alpha < 0$. Hence it is necessary that

$$\mathcal{B} < (1 + (1+8\alpha)^{1/2})/2.$$

The next application is an elementary proof of the Erdös-Rényi form of the Borel-Cantelli lemma [1, p. 326].

COROLLARY 2. *If* $A_k \in \mathfrak{F}$, $k = 1, 2, 3, \cdots$, *with* $\sum_{k=1}^{\infty} P(A_k) = +\infty$, then $P(\bigcup_{k=N}^{\infty} A_k) \geqq 1/c$ *where*

$$c = \liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \sum_{l=1}^{n} P(A_k \cap A_l)}{\left(\sum_{k=1}^{n} P(A_k)\right)^2}.$$

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PROOF. By Theorem 1.1

$$P\left(\bigcup_{k=N}^{M} A_k\right) \geqq \frac{\theta}{\dfrac{2\alpha_{NM}}{\mathcal{B}_{NM}^2} + \dfrac{(2-\theta)}{\mathcal{B}_{NM}}} + \frac{(1-\theta)}{\dfrac{2\alpha_{NM}}{\mathcal{B}_{NM}^2} + \dfrac{(1-\theta)}{\mathcal{B}_{NM}}}$$

where

$$\begin{aligned} 2\alpha_{NM} &= 2\sum_{k=N}^{M}\sum_{l=N}^{k-1} P(A_k \cap A_l) \\ &= \sum_{k=N}^{M}\sum_{l=N}^{M} P(A_k \cap A_l) - \mathcal{B}_{NM} \end{aligned}$$

and

$$\mathcal{B}_{NM} = \sum_{k=N}^{M} P(A_k).$$

Hence since $\lim_{M\to\infty}\mathcal{B}_{NM} = +\infty$,

$$\lim_{m\to\infty} P\left(\bigcup_{k=N}^{M} A_k\right) \geqq \frac{1}{c_N}$$

where

$$\begin{aligned} c_N &= \liminf_{M\to\infty} \frac{\sum_{k=N}^{M}\sum_{l=N}^{M} P(A_k \cap A_l)}{\mathcal{B}_{NM}^2} \\ &= c. \end{aligned}$$

Corollary 2 can also be proved by the following inequality due to Chung and Erdös [2],

$$P\left(\bigcup_{k=1}^{N} A_k\right) \geqq \frac{\mathcal{B}^2 - \mathcal{B}}{2\alpha}. \tag{1.2}$$

(1.2) is easily proved by using Schwarz's inequality applied to indicator functions of sets.

However the inequality of Theorem 1.1 is stronger than (1.2) unless $(\mathcal{B}^2 - \mathcal{B})/2\alpha = 1$.

In fact it is easy to verify that

$$\frac{\mathcal{B}^2}{2\alpha + \mathcal{B}} > \frac{\mathcal{B}^2 - \mathcal{B}}{2\alpha} \qquad \text{if } 0 \leqq \frac{\mathcal{B}^2}{2\alpha + \mathcal{B}} < 1$$

and

$$\frac{\mathcal{B}^2}{2\alpha+\mathcal{B}} = \frac{\mathcal{B}^2-\mathcal{B}}{2\alpha} \quad \text{if } \frac{\mathcal{B}^2}{2\alpha+\mathcal{B}} = 1.$$

It will be shown elsewhere that the lower bound of Theorem 1.1 can be attained so that the inequality cannot be improved.

2. **Proof of Theorem 1.1.** Given $\{A_1, \cdots, A_N\}$ we define B_r, $r=1, \cdots, N$, to be the set of points in $\bigcup_{k=1}^N A_k$ which belong to exactly r of the sets $\{A_1, \cdots, A_N\}$ and let $a_r = P(B_r)$.

Then we have

$$a_r \geqq 0, \qquad r = 1, \cdots, N, \tag{2.1}$$

$$\sum_{k=1}^N k a_k = \sum_{k=1}^N P(A_k) = \mathcal{B}, \tag{2.2}$$

and

$$\sum_{k=2}^N \frac{k(k-1)}{2} a_k = \sum_{k=1}^N \sum_{l=1}^{k-1} P(A_k \cap A_l) = \alpha. \tag{2.3}$$

We proceed by finding the minimum of the linear expression

$$V = P\left(\bigcup_{k=1}^N A_k\right) = \sum_{k=1}^N a_k \tag{2.4}$$

subject to the constraints (2.1), (2.2) and (2.3). Substituting the expression for a_{r-1} which is obtained by solving (2.2) for a_{r-1} into (2.3), and dividing by r to make the coefficient of a_r equal to one we obtain

$$\begin{aligned} \frac{-(r-2)}{r} a_1 + \sum_{k=2}^N a_k \left\{\frac{k(k-1)}{r} - \frac{k(r-2)}{r}\right\} \\ = \frac{2\alpha}{r} - \frac{(r-2)}{r}\mathcal{B}. \end{aligned} \tag{2.5}$$

Substituting the expression for a_r which is obtained by solving (2.3) for a_r into (2.2), we obtain

$$a_1 + \sum_{k=2}^N a_k \left\{k - \frac{k(k-1)}{r-1}\right\} = \mathcal{B} - \frac{2\alpha}{r-1}\cdot \tag{2.6}$$

Now solving (2.5) for a_r and (2.6) for a_{r-1} and substituting the resulting expressions into (2.4) we obtain

$$V - \frac{2\mathcal{B}}{r} + \frac{2\alpha}{r(r-1)} = a_1 \frac{(r-2)}{r} + \sum_{k=2}^N \frac{(r-k)(r-k-1)}{r(r-1)} a_k. \tag{2.7}$$

We henceforth assume that in equations (2.5), (2.6) and (2.7),

$$r = 2 + [2\alpha/\mathcal{B}] \quad \text{if } 2\alpha/\mathcal{B} \neq (N-1)$$
$$= N \quad \text{if } 2\alpha/\mathcal{B} = N-1. \tag{2.8}$$

Since $2\alpha \leqq (N-1)\mathcal{B}$ it follows that $r \leqq N$.

It then follows that

$$(r-2)/r \geqq 0 \tag{2.9}$$

and

$$(r-k)(r-k-1)/r(r-1) \geqq 0, \quad k = 2, \cdots, N. \tag{2.10}$$

Since (2.9) and (2.10) imply that the coefficients of the a_{κ} in the right-hand side of equation (2.7) are nonnegative, the minimal value of $V - 2\mathcal{B}/r + 2\alpha/r(r-1)$ is zero.

But if we set

$$a_{r-1} = \mathcal{B} - 2\alpha/(r-1),$$
$$a_r = 2\alpha/r - (r-2)\mathcal{B}/r, \tag{2.11}$$
$$a_k = 0, \quad k \neq r-1, r,$$

then equations (2.5) and (2.6) are satisfied and $V - 2\mathcal{B}/r + 2\alpha/r(r-1) = 0$. Moreover (2.8) implies that $a_{r-1} \geqq 0$ and $a_r \geqq 0$ so that the set of a_n's given by (2.11) minimizes the expression (2.4) and satisfies all the constraints.

Therefore the minimal value of V is

$$V = \frac{2\mathcal{B}}{r} - \frac{2\alpha}{r(r-1)}$$
$$= \frac{2\mathcal{B}^2}{2\alpha + (2-\theta)\mathcal{B}} - \frac{2\alpha\mathcal{B}^2}{(2\alpha + (2-\theta)\mathcal{B})(2\alpha + (1-\theta)\mathcal{B})}$$

where $\theta = 2\alpha/\mathcal{B} - [2\alpha/\mathcal{B}]$, $0 \leqq \theta < 1$. But then

$$V = \frac{\theta\mathcal{B}^2}{2\alpha + (2-\theta)\mathcal{B}} + \frac{(1-\theta)\mathcal{B}^2}{2\alpha + (1-\theta)\mathcal{B}}.$$

References

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2. K. L. Chung and P. Erdös, *On the application of the Borel-Cantelli lemma*, Trans. Amer. Math. Soc. **72** (1952), 179–186.

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