AN INEQUALITY FOR PROBABILITIES

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1. The main result and applications. Given a probability measure space \((\Omega, \mathcal{F}, P)\), let \(A_k \subseteq \Omega, k = 1, \ldots, N\). The main result is given in the following theorem.

**Theorem 1.1.**

\[
P\left( \bigcup_{k=1}^{N} A_k \right) \geq \frac{\theta \mathcal{B}^2}{2\alpha + (2 - \theta)\mathcal{B}} + \frac{(1 - \theta)\mathcal{B}^2}{2\alpha + (1 - \theta)\mathcal{B}}
\]

where \(\mathcal{B} = \sum_{k=1}^{N} P(A_k), \alpha = \sum_{k=1}^{N} \sum_{l=1}^{k-1} P(A_k \cap A_l)\) and \(\theta = 2\alpha/\mathcal{B} - [2\alpha/\mathcal{B}], 0 \leq \theta < 1\).

The proof of Theorem 1.1 is given in §2.

**Corollary 1.** A necessary condition for \(P\left( \bigcup_{k=1}^{N} A_k \right) < 1\) is that \(\mathcal{B} < (1 + (1 + 8\alpha)^{1/2})/2\).

**Proof.** It is easy to verify that if the right-hand side of inequality (1.1) is regarded as a function of \(\theta\), then the minimum occurs for \(\theta = 0\). Hence

\[
P\left( \bigcup_{k=1}^{N} A_k \right) \geq \frac{\mathcal{B}^2}{2\alpha + \mathcal{B}}.
\]

Therefore if \(1 > P\left( \bigcup_{k=1}^{N} A_k \right)\), then \(1 > \mathcal{B}^2/(2\alpha + \mathcal{B})\), that is, \(\mathcal{B}^2 - \mathcal{B} - 2\alpha < 0\). Hence it is necessary that \(\mathcal{B} < (1 + (1 + 8\alpha)^{1/2})/2\).

The next application is an elementary proof of the Erdős-Rényi form of the Borel-Cantelli lemma [1, p. 326].

**Corollary 2.** If \(A_n \subseteq \Omega, k = 1, 2, 3, \ldots\), with \(\sum_{k=1}^{\infty} P(A_k) = +\infty\), then \(P\left( \bigcup_{k=1}^{n} A_k \right) \geq 1/c\) where

\[
c = \liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \sum_{l=1}^{n} P(A_k \cap A_l)}{\left( \sum_{k=1}^{n} P(A_k) \right)^2}.
\]

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PROOF. By Theorem 1.1

\[ P\left( \bigcup_{k=N}^{M} A_k \right) \geq \frac{\theta}{2\alpha_{NM}} + \frac{(1 - \theta)}{\beta_{NM}} \]

where

\[ 2\alpha_{NM} = 2 \sum_{k=N}^{M} \sum_{l=N}^{k-1} P(A_k \cap A_l) \]

\[ = \sum_{k=N}^{M} \sum_{l=N}^{k-1} P(A_k \cap A_l) - \beta_{NM} \]

and

\[ \beta_{NM} = \sum_{k=N}^{M} P(A_k). \]

Hence since \( \lim_{M \to \infty} \beta_{NM} = +\infty \).

\[ \lim_{n \to \infty} P\left( \bigcup_{k=N}^{M} A_k \right) \geq \frac{1}{c_N} \]

where

\[ c_N = \liminf_{M \to \infty} \frac{\sum_{k=N}^{M} \sum_{l=N}^{k-1} P(A_k \cap A_l)}{\beta_{NM}^2} \]

\[ = c. \]

Corollary 2 can also be proved by the following inequality due to Chung and Erdös [2],

\[ P\left( \bigcup_{k=1}^{N} A_k \right) \geq \frac{\beta^2 - \beta}{2\alpha} \cdot \]

(1.2) is easily proved by using Schwarz's inequality applied to indicator functions of sets.

However the inequality of Theorem 1.1 is stronger than (1.2) unless \((\beta^2 - \beta)/2\alpha = 1\).

In fact it is easy to verify that

\[ \frac{\beta^2}{2\alpha + \beta} > \frac{\beta^2 - \beta}{2\alpha} \quad \text{if} \quad 0 \leq \frac{\beta^2}{2\alpha + \beta} < 1 \]

and
It will be shown elsewhere that the lower bound of Theorem 1.1 can be attained so that the inequality cannot be improved.

2. Proof of Theorem 1.1. Given \( \{A_1, \ldots, A_N\} \) we define \( B_r, r=1, \ldots, N, \) to be the set of points in \( \bigcup_{k=1}^{N} A_k \) which belong to exactly \( r \) of the sets \( \{A_1, \ldots, A_N\} \) and let \( a_r=P(B_r) \).

Then we have

\[
(2.1) \quad a_r \geq 0, \quad r = 1, \ldots, N,
\]

\[
(2.2) \quad \sum_{k=1}^{N} k a_k = \sum_{k=1}^{N} P(A_k) = \mathcal{B},
\]

and

\[
(2.3) \quad \sum_{k=2}^{N} \frac{k(k-1)}{2} a_k = \sum_{k=1}^{N} \sum_{l=1}^{k-1} P(A_k \cap A_l) = \alpha.
\]

We proceed by finding the minimum of the linear expression

\[
(2.4) \quad V = P\left( \bigcup_{k=1}^{N} A_k \right) = \sum_{k=1}^{N} a_k
\]

subject to the constraints (2.1), (2.2) and (2.3). Substituting the expression for \( a_{r-1} \) which is obtained by solving (2.2) for \( a_{r-1} \) into (2.3), and dividing by \( r \) to make the coefficient of \( a_r \) equal to one we obtain

\[
(2.5) \quad \frac{-(r-2)}{r} a_1 + \sum_{k=2}^{N} a_k \left\{ \frac{k(k-1)}{r} - \frac{k(r-2)}{r} \right\} = \frac{2\alpha}{r} - \frac{(r-2)}{r} \mathcal{B}.
\]

Substituting the expression for \( a_r \) which is obtained by solving (2.3) for \( a_r \) into (2.2), we obtain

\[
(2.6) \quad a_1 + \sum_{k=2}^{N} a_k \left\{ k - \frac{k(k-1)}{r-1} \right\} = \mathcal{B} - \frac{2\alpha}{r-1}.
\]

Now solving (2.5) for \( a_r \) and (2.6) for \( a_{r-1} \) and substituting the resulting expressions into (2.4) we obtain

\[
(2.7) \quad V = \frac{2\mathcal{B}}{r} + \frac{2\alpha}{r(r-1)} = a_1 \frac{(r-2)}{r} + \sum_{k=2}^{N} \frac{(r-k)(r-k-1)}{r(r-1)} a_k.
\]
We henceforth assume that in equations (2.5), (2.6) and (2.7),

\[
(2.8) \quad r = 2 + \left[ \frac{2\alpha}{\mathfrak{B}} \right] \quad \text{if} \quad 2\alpha/\mathfrak{B} \neq (N - 1) \\
= N \quad \text{if} \quad 2\alpha/\mathfrak{B} = N - 1.
\]

Since \(2\alpha \leq (N - 1)\mathfrak{B}\) it follows that \(r \leq N\).

It then follows that

\[
(2.9) \quad \frac{(r - 2)}{r} \geq 0
\]

and

\[
(2.10) \quad \frac{(r - k)(r - k - 1)}{r(r - 1)} \geq 0, \quad k = 2, \ldots, N.
\]

Since (2.9) and (2.10) imply that the coefficients of the \(a_n\) in the right-hand side of equation (2.7) are nonnegative, the minimal value of \(V - 2\mathfrak{B}/r + 2\alpha/r(r - 1)\) is zero.

But if we set

\[
\begin{align*}
a_{r-1} &= \mathfrak{B} - 2\alpha/(r - 1), \\
a_r &= 2\alpha/r - (r - 2)\mathfrak{B}/r, \\
a_k &= 0, \quad k \neq r - 1, r,
\end{align*}
\]

then equations (2.5) and (2.6) are satisfied and \(V - 2\mathfrak{B}/r + 2\alpha/r(r - 1) = 0\). Moreover (2.8) implies that \(a_{r-1} \geq 0\) and \(a_r \geq 0\) so that the set of \(a_n\)'s given by (2.11) minimizes the expression (2.4) and satisfies all the constraints.

Therefore the minimal value of \(V\) is

\[
V = \frac{2\mathfrak{B}}{r} - \frac{2\alpha}{r(r - 1)}
\]

\[
= \frac{2\mathfrak{B}^2}{2\alpha + (2 - \theta)\mathfrak{B}} - \frac{2\alpha\mathfrak{B}^2}{(2\alpha + (2 - \theta)\mathfrak{B})(2\alpha + (1 - \theta)\mathfrak{B})}
\]

where \(\theta = 2\alpha/\mathfrak{B} - \left[ \frac{2\alpha}{\mathfrak{B}} \right], 0 \leq \theta < 1\). But then

\[
V = \frac{\theta\mathfrak{B}^2}{2\alpha + (2 - \theta)\mathfrak{B}} + \frac{(1 - \theta)\mathfrak{B}^2}{2\alpha + (1 - \theta)\mathfrak{B}}.
\]

REFERENCES


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