DAVID SANKOFF

PARTY STRATEGY AND THE RELATIONSHIP BETWEEN VOTES AND SEATS

Game theoretical treatments of electoral campaigns have modeled either one of two aspects of the link between campaign strategy and voting behaviour. The so-called spatial theory assumes (somewhat cynically) that candidates are free to move in an issue space in search of an optimal position with respect to an electorate. The latter is represented by a probability distribution over this space and a voter occupying a given position is assumed to vote for the nearest candidate. A second approach (even more cynical) ignores issues entirely and assumes that in a given area, differential voter response to candidates is a function only of differential expenditure of resources, financial or otherwise, by the candidates, in that area. In the latter type of analysis, the nature of the electoral law and representational system plays a central role. This paper summarizes and compares recent work on resource allocation games and develops the mathematical apparatus for studying more realistic models than have been considered to date. A by-product of this approach is a new type of explanation for vote-seat relationships such as the cube law (Kendall and Stuart, 1950).

I. GAMES WITH COMPLETE SECRECY

A key component of real electoral strategy is secrecy about resource allocation. Money and effort spent in a given area will be more effective if the opponent does not find out about it beforehand or soon enough afterward to compensate. This aspect of campaign strategy can be modeled by a type of game originally known as 'Blotto'.

There are two opponents, I and II, with known resources R_{I} and R_{II} respectively. Each player i allocates (secretly) his resources among n contests, i.e. chooses a vector $(r_1^i, \ldots, r_n^i)$ such that $r_k^i \geqslant 0$ and $\sum_k r_k^i = R_i$. If $r_k^{\mathrm{I}} > r_k^{\mathrm{II}}$ we say that I wins contest k, and if $r_k^{\mathrm{I}} < r_k^{\mathrm{II}}$, we say II wins. Each player must find a (mixed) strategy which maximizes the minimum number of contests (i.e. constituencies, districts, or counties) he can expect to win over all possible strategies played by his opponent.

Theory and Decision **5** (1974) 289–294. *All Rights Reserved*

This game has a solution. One set of nearly optimal strategies can be visualized as follows. Suppose $R_{\mathrm{I}} > R_{\mathrm{II}}$. Then player I chooses an allocation using the following procedure. He arranges the n contests in a random order and equally spaced along the unit interval so that x_k is the position of the kth contest. He constructs the linear function $r(x) = 2R_{\mathrm{I}}(1-x)/n$ and allocates $r_k^{\mathrm{I}} = r(x_k)$ to the kth contest. Player II randomizes the contests as well, but assigns no resources to the kth contest if $x_k > R_{\mathrm{II}}/R_{\mathrm{I}}$. On the interval $(0, R_{\mathrm{II}}/R_{\mathrm{I}})$ he allocates resources according to the function $r(x) = 2R_{\mathrm{I}}(1 - xR_{\mathrm{I}}/R_{\mathrm{II}})/n$. By ignoring some contests, the player with lesser resources is able to compete on an equal footing with the richer player in the rest. The value of the game, or the proportion of contests player I can expect to win, is $1 - R_{\mathrm{II}}/2R_{\mathrm{I}}$. Proofs can be found in Sankoff and Mellos (1972) and, using other types of optimal strategies, in a series of Rand corporation memoranda of 1950–51, by Gross and Wagner, and in a paper of Friedman (1958).

In applying this model to the single representative, simple plurality electoral system, what does the game value tell us about the relationship between V_{I}, the proportion of the total popular vote obtained by party I, and S_{I}, the proportion of seats he expects to win? To answer this, we must specify the association between resources spent r_k^i and votes obtained v_k^i within an electoral district k. An easy, through somewhat unrealistic, way of doing this is through simple proportionality: $v_k^i = cr_k^i$. Then $S_{\mathrm{I}} = 1 - R_{\mathrm{II}}/2R_{\mathrm{I}} = 1 - \sum v_k^{\mathrm{II}}/2\sum v_k^{\mathrm{I}} = 1 - V_{\mathrm{II}}/2V_{\mathrm{I}} = (3V_{\mathrm{I}} - 1)/2V_{\mathrm{I}}$. The slope of S_{I} as a function of V_{I} is 2 at the point $(\frac{1}{2}, \frac{1}{2})$. This slope, which we call the swing ratio, is halfway between the value for proportional representation on one hand, and the value for the cube law (i.e. 3) on the other.

A more realistic association between resources and votes might be $v_k^i = c_0 + cr_k^i$, where c_0 represents equal hard-core contributions to the support of both parties in each district. This is still overly simple but it is an improvement on the basic model and it requires the same game theory solution. I.e. once again $S_{\mathrm{I}} = 1 - R_{\mathrm{II}}/2R_{\mathrm{I}}$, but this now leads to a swing ratio of $2/(1-2nc_0)$. So if $\frac{1}{3}$ of all voters are hard-core, we arrive at a swing ratio of 3, if $\frac{1}{2}$ are hard-core, we get a swing ratio of 4, and so on.

II. GAMES WITH PERFECT INFORMATION

The model based on the complete secrecy assumption is obviously unrealistic. At the other extreme, where both players know exactly what the other one is doing, this type of game has no solution, since there are no optimal pure strategies (unless R_{II} is trivially small). Brams and Davis (1974) have introduced another type of game with perfect information, where voting within a district is still a function of resources spent, but a function which incorporates a random component. This game has equilibrium strategies when $R_I = R_{II}$, but little else is known about it.

III. COMBINING BOUND AND FREE RESOURCES

Despite Watergate, games with perfect information are about as unrealistic as games with perfect secrecy. I would like to suggest an approach to controlling the balance between information and secrecy in allocation games. Let us return to our original game, but in addition to the freely allocatable resources R_I and R_{II}, we suppose that in each district k, the parties already have some bound (or fixed) resources, say f_k^I and f_k^{II}, respectively, and that the f_k^i are known to both parties. Party I wins if $r_k^I + f_k^I > r_k^{II} + f_k^{II}$ and II wins if the inequality is reversed. If the known, bound resources are large compared to the R_i, then the game approaches perfect information. If the bound resources are very small, on the other hand, then the game approaches perfect secrecy. This game may also be considered a further generalization of the secret game mentioned at the end of Section I above, where the parties had equal numbers of hard-core supporters in each district k. In the present game the f_k^i do not all have to be equal.

Though it would be preferable to solve this game in full generality, the solutions presently available for certain special cases provide a great deal of insight into the types of optimal strategies which arise.

IV. A GAME WITH A FIXED ADVANTAGE IN ALL DISTRICTS

For example, suppose $f_k^{II} - f_k^I = f > 0$ for all k. This might be a model for a situation where one party, perhaps the one presently in power, has a systematic advantage in all districts.

To solve this game, we extend the basic principle that, on a set of the districts where both parties allocate free resources, an optimal strategy for the party with fewer total resources to spend in that set is to ignore some districts so that it can compete on an equal footing in the others. This principle can be derived from the fact that the swing ratio is greater than one in the purely secret game.

Choose resource units so that $R_{\mathrm{I}}+R_{\mathrm{II}}+F=1$, where $F=nf$. Then if party I competes on a proportion D_{I} of the districts, and II on D_{II}, the principle requires that $R_{\mathrm{II}}/D_{\mathrm{II}}=(R_{\mathrm{I}}-FD_{\mathrm{I}})/D_{\mathrm{I}}$. Since the two parties allocate independently, the subsets of size D_{I} and D_{II} will be chosen independently and I will win a proportion $D_{\mathrm{I}}(1-D_{\mathrm{II}})+\frac{1}{2}D_{\mathrm{I}}D_{\mathrm{II}}$ of the districts. Maximizing with respect to D_{I}, we find that if $F>R_{\mathrm{I}}-\frac{1}{4}R_{\mathrm{II}}(1+\sqrt{1+8R_{\mathrm{I}}/R_{\mathrm{II}}})$, then $D_{\mathrm{I}}<1$, and party I wins $(R_{\mathrm{I}}/F)\,[1+(R_{\mathrm{II}}/F)(1-\sqrt{1+2F/R_{\mathrm{II}}})]$. Otherwise, $D_{\mathrm{I}}=1$ and the game becomes identical to our original game with resources $R_{\mathrm{I}}-F$ and R_{II}, respectively, so that I wins $1-R_{\mathrm{II}}/2(R_{\mathrm{I}}-F)$.

What does all this mean in terms of seats and votes? For comparison's sake, we will work in terms of the simple proportionality between resources and votes, though hard-core considerations are just as easy to incorporate as in the purely secret game. For small F, say $F=R_{\mathrm{II}}/9$, the seat vote curve is almost indistinguishable from that of the purely secret game. But as F becomes a greater proportion of II's resources, the curve changes in character. I's share of the seats increases more rapidly as R_{I} moves from zero upward, so that if $F=9R_{\mathrm{II}}$, say, I can expect to win 50% of the seats with only 42% of the votes. With 50% of the votes, he can expect 70% of the seats. Thus we have found a framework for assessing the relative value of bound versus free resources.

Another feature of these curves is that if we assess the swing ratio at $S_{\mathrm{I}}=0.5$ and not at $V_{\mathrm{I}}=0.5$, this ratio remains 2. (This has not been proved, only observed in a number of calculated curves.)

V. A GAME WITH INCUMBENT ADVANTAGE IN ALL DISTRICTS

The final game to be discussed is a generalization of the previous one. Here $f_k^{\mathrm{II}}-f_k^{\mathrm{I}}=f$ for a proportion M of the districts, and $f_k^{\mathrm{I}}-f_k^{\mathrm{II}}=f$ for $L=1-M$ of them. This might be a model of the situation where, in a preceding election, party I won a proportion L of the seats and II won M, and where incumbency results in a fixed advantage in all districts.

Of course, this game is much less tractable than the previous one. Nevertheless some interesting results are obtainable.

Let $F = nf$ as before and $R_{\mathrm{I}} + R_{\mathrm{II}} + (M+L)\,F = R_{\mathrm{I}} + R_{\mathrm{II}} + F = 1$. Suppose party I assigns aR_{I} to the set of seats where it has the advantage, and $(1-a)\,R_{\mathrm{I}}$ to the others, while II assigns $(1-b)\,R_{\mathrm{II}}$ and bR_{II}, respectively. Is it possible that for some optimal strategies, a and b are fixed proportions? *A priori*, we cannot assume they are, since an optimal mixed strategy for I, say, might very well require that a be picked according to some probability distribution. If a and b *could* be constants, however, it would facilitate the solution of the game, for then we could decompose the game into two games of the previous type, one where I has the advantage in all districts, and the other one where II has the advantage. For a and b to be constant, it would be necessary for S_{I} to have a saddle point as a function of a and b. It is possible to show that the existence of a saddle point is equivalent to the existence of appropriate solutions of certain equations. We can solve these equations numerically for given R_{I}, R_{II}, M, L, and F, and hence show that a saddle point exists. Thus optimal values (constants) for a and b can be found, which considerably mitigates the difficulties inherent in this game.

The relationship between resources and votes has not yet been incorporated in this model, mainly because the seat-resource relationships are not available as formulae, but must be computed numerically. In the same way, however, numerical tabulation of seat-vote curves is feasible.

In summary, I have sketched the extent of current knowledge about a class of resource allocation games, some of which are considerably more realistic than the original Blotto game. This research indicates that the departure from proportionality in simple plurality single representative systems can be explained in large part through the strategic problems parties encounter by virtue of the game-like character of this electoral system.

Future work on these problems will aim at the solution of the game with arbitrary patterns of bound and free resources. In addition, an attempt must be made to come to terms with the non-uniqueness of optimal strategies. One way to do this is, for given strategies, to compare the expected distribution of $v_k^{\mathrm{I}}/(v_k^{\mathrm{I}} + v_k^{\mathrm{II}})$ across all districts k, with the distributions observed in real electoral systems by Kendall and Stuart (1950), March (1957–8), and Sankoff and Mellos (1973).

Centre de recherches mathématiques,
Université de Montreal

REFERENCES

S. Brams and M. D. Davis, 'The 3/2's Rule in Presidential Campaigning', *Amer. Pol. Sci. Rev.* **LXVIII** (1974) 113–34.

L. Friedman, 'Game Theory Models in the Allocation of Advertising Expenditures', *Operations Research* 6 (1958) 699–709.

O. Gross and R. Wagner, 'A Continuous Colonel Blotto Game', Rand Memorandum 408 (1950).

O. Gross, 'The Symmetric Blotto Game', Rand Memorandum 424 (1950).

O. Gross, 'An Infinite-Dimensional Extension of a Symmetric Blotto Game', Rand Memorandum 718 (1951).

M. G. Kendall and A. Stuart, 'The Law of the Cubic Proportion in Election Results', *British J. Soc.* **1** (1950), 183–197.

J. G. March, 'Party Legislature Representation as a Function of Election Results', *Public Opinion Quarterly* **XXI** (1957–58) 521–42.

D. Sankoff and K. Mellos, 'The Swing Ratio and Game Theory', *Amer. Pol. Sci. Rev.* **LXVI** (1972) 551–554.

D. Sankoff and K. Mellos, 'La régionalisation électorale et l'amplification des proportions', *Revue canadienne de science politique* **VI** (1973) 380–98.