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GRAVITY MODELS AND INTERACTION PROBABILITIES*

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1. INTRODUCTION

Many gravity expressions have been proposed to describe human interactions and many theoretical justifications for them have been suggested. However, for various reasons, the link between gravity expressions and the probability of interaction at a distance was long neglected. In 1964, Harris [2] pointed out this deficiency of literature in the following terms: "There is no clear statement in the literature as to the probability density function for the distribution of interaction over distance, and such a probability density function is difficult or impossible to formulate for this (gravity) model in its original and most widely used form." During the last fifteen years, however, several authors, such as Schneider [4], Harris [2, 3], and Choukroun [1], have used probability theory for studying gravity models. Harris [2] used it for reconciling the intervening opportunities theory and the traditional inverse-distance model. Choukroun [1], following a similar approach, formulated a model based on the multinomial distribution to generate a negative exponential interaction function. These contributions largely pertain either to specific gravity hypotheses or to specific distribution functions though the macromodel proposed by Choukroun is of a very general nature.

This paper attempts to provide further insight into the behavior-theoretic content of gravity hypotheses by focusing on the relation between gravity hypotheses and the concept of interaction probability. This is done by deriving, for all gravity hypotheses satisfying certain conditions, three probability functions expressing different aspects of the interaction phenomenon. These functions are (1) the *interaction probability density*, which yields the probability that any given interaction will involve a distance in a specified interval, (2) the *interaction decision function* (and the associated *threshold density*) which gives the probability of a generator deciding to interact when considering an attractor at a given distance and (3) the *density of attractor distances*. These three probabilistic concepts are then explored in the context of the inverse-distance and negative-exponential forms of human gravity.

2. INTERACTION PROBABILITIES

We shall model a system of human interaction in terms of a probability space whose events can be interpreted as having two components, namely a choice of the

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distance **d** over which an interaction is considered, and secondly the actual decision to interact or not to interact. The outcome of this decision will be denoted by I or $\sim I$, as the case may be. Other events in the space to be considered are of the form $(I, \mathbf{d} \leqslant x) = I \cap \mathbf{d} \leqslant x$, $(\sim I, \mathbf{d} \leqslant x)$, $\mathbf{d} \leqslant x$, $\mathbf{d} = x$, etc., where x is a nonnegative real number.

For any suitably differentiable probability P on this set of events, we define three functions on $[0, \infty]$—the *interaction density*

$$(1) \qquad f(x) = d/dx[P(\mathbf{d} \leqslant x \mid I)],$$

the *interaction decision function*

$$(2) \qquad g(x) = -d/dx[P(I \mid \mathbf{d} = x)],$$

and the *density of attractor distances*

$$(3) \qquad a(x) = d/dx[P(\mathbf{d} \leqslant x)].$$

Examining the conditional probability in (2), we have

$$(4) \qquad P(I \mid d = x) = \frac{d/dx[P(I, \mathbf{d} \leqslant x)]}{d/dx[P(\mathbf{d} \leqslant x)]} = P(I)[f(x)/a(x)],$$

wherever this quotient exists, from which

$$(5) \qquad g(x) = -P(I)d/dx[f(x)/a(x)].$$

Generally, the observed deterrent effect of distance on interaction is consistent with a concentration of the interaction density on short distances. We make a mild assumption to this effect, namely that the interaction density tails off more rapidly than the density of attractors, or

$$(6) \qquad \lim_{x\to\infty} [f(x)/a(x)] = 0$$

so that

$$(7) \qquad \int_x^\infty g(y)dy = P(I)[f(x)/a(x)]$$

Thus for fixed $P(I)$ and a, any interaction density f satisfying (6) uniquely determines the interaction decision function g and is uniquely determined by g.

An interesting feature of (7) is the relation which it defines between three aspects of the interaction phenomenon, namely the observable, the behavioral, and the environmental aspects. The observable aspect is characterized by the interaction probability density f which corresponds to the outcome of various interaction decisions made in the system. By contrast, the interaction decision function g characterizes the attitudes of decision-makers with respect to distance. It is a behavioral concept in that it does not intrinsically depend on the environment of decision-makers. It actually determines their reaction to their environment. In this model, it is the density of attractor distances, a, which describes the environment in terms of distances between decision-makers and attractors. The density a may also be thought of as having a behavioral component in that it may represent, in practice, the density of attractors *perceived* by the decision-maker.

3. THE THRESHOLD DENSITY

In studying the deterrent effect due solely to distance on interaction, we may postulate $P(I \mid \mathbf{d} = 0) = 1$ so that from (4) and continuity

$$P(I) = \lim_{x \searrow 0} [a(x)/f(x)] \leqslant 1 \tag{8}$$

and

$$\int_0^\infty g(y)dy = 1 \tag{9}$$

from (7). Under the somewhat more restrictive condition that f/a be monotone nonincreasing, we also have, from (5), that g is nonnegative and, hence, a probability density on $[0, \infty]$.

For this situation, an intuitively appealing interpretation of g is suggested by Dr. T. E. Smith. Consider two independent random variables $\mathbf{d}'$ and $\mathbf{t}$, with densities a and g, respectively, defined on $[0, \infty]$. Letting P' be the joint probability function and I' be the event that $\mathbf{d}' \leqslant \mathbf{t}$, it follows that for all x in $[0, \infty]$,

$$\begin{aligned} d/dx[P'(I', \mathbf{d}' \leqslant x)] &= a(x) \int_x^\infty g(y)dy \\ &= P(I)f(x) \\ &= d/dx[P(I, \mathbf{d} \leqslant x)] \end{aligned} \tag{10}$$

by (7) and (1), so that this new model is identical to the one we have been considering. In the new interpretation, $\mathbf{t}$ is a random threshold which is sampled before $\mathbf{d}'$, the random attractor distance. Whenever the attractor distance is below the threshold, interaction always occurs, and only then.

The threshold concept is of interest both from the theoretical and empirical points of view. From a theoretical standpoint, the requirement that g be a density in order to be interpreted as a threshold density, can be viewed as an argument for requiring f/a to be monotone nonincreasing. If the threshold process is really to describe human behavior, then it follows that f/a must be monotone nonincreasing. This has another implication. It will be shown further on that, under certain conditions, it is possible to relate each interaction probability density to a specific form of gravity. Then the restriction imposed on f/a by the adoption of a threshold framework leads to rejecting all the gravity expressions for which f/a is not monotone nonincreasing. On the other hand, this notion of threshold lends itself to empirical analysis. It has a concrete meaning which, from an empirical point of view, may make it more valuable than, for instance, the concept of utility.

4. GRAVITY EXPRESSIONS AND THE INTERACTION DENSITIES

The interaction model we have defined may be interpreted in terms of the concept of human gravity. This concept relates some measure i_{uv} of interaction between points u and v in a given metric space, to the quotient of two quantities, the numerator related to the masses or mass densities associated with the two points, and the denominator an increasing function γ of the distance $d(u, v)$ between the two points. The numerator incorporates the positive effect of increasing mass on the rate or intensity of interaction, while the deterrence function, γ, expresses the dissuading effect of distance on interaction. In concentrating on the latter, we will assume a uniform mass distribution over our space so that we may write

$$i_{uv} = 1/\gamma[d(u, v)] \tag{11}$$

Since we are specifically interested in the effects of distance, we will assume that the density of attractor distances a is the same for all generators, and writing $i(x)$ for the relative intensity or rate of interaction over a distance x, we arrive at

$$i(x) = a(x)/\gamma(x). \tag{12}$$

In our probabilistic interaction model, the corresponding density of the interaction probability is $d/dx[P(I, \mathbf{d} \leqslant x)]$ which, as in (10), is equal to $P(I)f(x)$. Thus, the identification of the two formulations leads to

$$\begin{aligned} P(I)f(x) &= i(x) \\ &= a(x)/\gamma(x) \end{aligned} \tag{13}$$

when i and γ are expressed in appropriate units. Then from (7) and (13),

$$\int_x^\infty g(y)dy = 1/\gamma(x) \tag{14}$$

Equations (7) and (14) clearly define the relation which exists in the model between any deterrence function and the observable, behavioral, and environmental aspects of the corresponding interaction system.

5. THE INVERSE-DISTANCE HYPOTHESIS IN THE TWO-DIMENSIONAL CASE

One way to study and compare gravity expressions is to find densities, f, g and a, which correspond to their deterrence functions. The traditional inverse-distance hypothesis in which $\gamma(x) = x$ does not correspond to any density g. Hence it has been modified and generalized in the *rational* hypothesis, in which the deterrence function may be written

$$\gamma_{r,c}(x) = (1 + cx)^r \tag{15}$$

where r and c are positive constants. From (14)

$$g_{r,c}(x) = cr/(1 + cx)^{r+1} \tag{16}$$

for all $x \geqslant 0$, and this Pareto distribution is indeed a well-defined threshold density such as is discussed in Section 3 above.

In a two-dimensional geographical model, we may consider attractors to be uniformly distributed in some neighborhood of the generator, so that the density, $a(x)$, of attractor distances will be proportional to the circumference of a circle of radius x, and hence will be proportional to the distance x itself, in this neighborhood. Then

$$f_{r,c}(x) \propto x/(1 + cx)^r \tag{17}$$

in this neighborhood, and condition (8) will be satisfied. The proportionality in (17) is interesting in that the interaction density actually increases with the distance x, at least within a restricted range of values of x, even when $r > 1$.

Note that an assumption $a(x) \propto x$ globally would preclude the probability density interpretation (3) of a, but is consistent with a probability density interpretation of $f_{r,c}$, as long as $r > 2$.

6. THE NEGATIVE EXPONENTIAL HYPOTHESIS

Consider the deterrence function

$$\tilde{\gamma}_r(x) = e^{rx} \tag{18}$$

for some $r > 0$. From (14),

$$\tilde{g}_r(x) = re^{-rx} \tag{19}$$

for all $x \geqslant 0$, this being the density of the negative exponential distribution with parameter r.

If $a(x) \propto x$ in a neighborhood of $x = 0$, then

$$\tilde{f}_r(x) \propto xe^{-rx} \tag{20}$$

in this neighborhood, which like some cases of (17), is an increasing function of x near zero. Again, allowing a to be the density of a nonprobability measure, such that $a(x) \propto x$ globally, leads to a probabilistic interpretation for $\tilde{f}_r$.

7. CONCLUSION

The simple model that has been presented here is based on the assumption that the choice of the distance over which an interaction is considered and the actual decision to interact or not to interact are two distinct though dependent events. This assumption has allowed us to distinguish the observable, the behavioral, and the environmental aspects of an interaction system and to relate these aspects to gravity models. This result provides a tool to analyze and compare existing gravity hypotheses. It could also lead to the formulation of new gravity hypotheses. Finally, it suggests new empirical approaches to gravity models since each function that has been defined in relation with gravity hypotheses can be empirically determined. From this point of view, the behavioral nature of the threshold density is of particular interest.

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